Lecture 3 - Lorentz Transformations

A Puzzle...

Example

A ruler is positioned perpendicular to a wall. A stick of length L flies by at speed v. It travels in front of the ruler, so that it obscures part of the ruler from your view. When the stick hits the wall it stops.

In your reference frame, the stick is shorter than L. Therefore, right before it hits the wall, you will be able to see a mark on the ruler that is less than L units from the wall.

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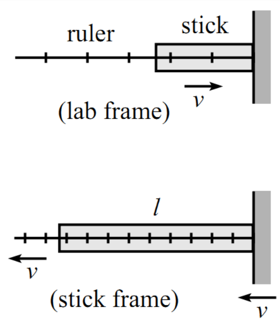

But in the stick's frame, the marks on the ruler are closer together. Therefore, when the wall hits the stick, the closest mark to the wall that you can see on the ruler is greater than L units.

Which view is correct (and what is wrong with the incorrect one)?

Solution

The first reasoning is correct. You will be able to see a mark on the ruler that is less than L units from the wall. (In fact, you will actually be able to see a mark even closer to the wall than $\frac{L}{\gamma}$, as we'll show below). The main point of this problem (and many other ones) is that signals do not travel instantaneously. The back of the stick does not know that the front of the stick has come into contact with the wall until a finite time has passed. Let's be quantitative about this and calculate (in both frames) the closest mark to the wall that you can see.

Consider the lab reference frame. The stick has length $\frac{L}{\gamma}$. Therefore, when the stick hits the wall, you can see mark a distance $\frac{L}{\gamma}$ from the wall. You will, however, be able to see a mark even closer to the wall, because the back end of the stick will keep moving forward, since it doesn't yet know that the front end has hit the wall. The stopping signal (shock wave, etc.) takes time to travel.

Let's assume that the stopping signal travels along the stick at speed c. (You can work with a general speed u. But the speed c is simpler, and it yields an upper bound on the closest mark you can see.) Where will the signal reach the back end? Starting from the time the stick hits the wall, the signal travels backwards from the wall at speed c, and the back end of the stick travels forwards at speed v (from a point $\frac{L}{\gamma}$ away from the wall). So the relative speed (as viewed by you) of the signal and the back end is $c+v$. Therefore, the signal hits the back end after a time $\frac{L/\gamma}{c+v}$. During this time, the signal has traveled a distance $\frac{c\,L/\gamma}{c+v}$ from the wall. The closest point to the wall that you can see is therefore the

$$\frac{L}{\gamma\left(1+\frac{v}{c}\right)} = \frac{L}{\gamma\,(1+\beta)} = L\left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tag{1}$$

mark on the ruler where we have used the notation

$$\beta \equiv \frac{v}{c} \tag{2}$$

Now consider the stick's reference frame. The wall is moving to the left towards it at speed v. After the wall hits the right end of the stick, the signal moves to the left with speed c, and the wall keeps moving to the left with speed v. Where is the wall when the signal reaches the left end? The wall travels $\frac{v}{c}$ as fast as the signal, so it travels a distance $\frac{L\,v}{c}$ in the time that the signal travels the distance L. This means that it is $L\left(1-\frac{v}{c}\right)$ away from the left end of the stick. In the stick's frame, this corresponds to a distance $\gamma\,L\left(1-\frac{v}{c}\right)$ on the ruler (because the ruler is length-contracted). So the left end of the stick is at the

$$\gamma\,L\,(1-\beta) = L\left(\frac{1-\beta}{1+\beta}\right)^{1/2} \tag{3}$$

mark on the ruler, in agreement with the result we found above. □

Lorentz Transformation

Introduction

Consider a coordinate system, S', moving relative to another system, S. Let the constant relative speed of the frames be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x-axis of S, in the positive direction. Nothing exciting happens in the y and z directions (since there is no transverse length contraction), so we'll mostly ignore them.

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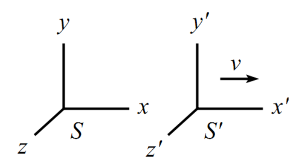

Our goal is to look at two events (an event is simply anything that has space and time coordinates) in space-time and relate the Δx and Δt of the coordinates in one frame to the $\Delta x'$ and $\Delta t'$ of the coordinates in another. We therefore want to find the constants A, B, C, and D in the relations,

$$\begin{aligned}\Delta x &= A\,\Delta x' + B\,\Delta t' \\ \Delta t &= C\,\Delta t' + D\,\Delta x'\end{aligned} \tag{4}$$

The four constants here will end up depending on v (which is constant, given the two inertial frames). We have assumed in the above equation that Δx and Δt are linear functions of $\Delta x'$ and $\Delta t'$. And we have also assumed that A, B, C, and D are constants (that is, dependent only on v, and not on x, t, x', t').

The first of these assumptions is justified by the fact that any finite interval can be built up from a series of many infinitesimal ones. But for an infinitesimal interval, any terms such as, for example, $(\Delta t)^2$, are negligible compared to the linear terms. Therefore, if we add up all the infinitesimal intervals to obtain a finite one, we will be left with only the linear terms.

Equivalently, suppose everyone decided that instead of meter sticks and seconds, we should all use half-meter sticks and half-seconds. In such a world, although all lengths and times would be doubled, speeds would remain the same and we would expect that all of the relativity results we have seen so far will be the same (because of the postulate that space is homogeneous). However, if there were any non-linear terms in the Lorentz transformations, we would be able to distinguish these two scenarios. For example, $\Delta x = A\,\Delta x' + B\,\Delta t' + E\,\Delta x'\,\Delta t'$ using meters and seconds would become $2\,\Delta\tilde{x} = 2\,A\,\Delta\tilde{x}' + 2\,B\,\Delta\tilde{t}' + 4\,E\,\Delta\tilde{x}'\,\Delta\tilde{t}'$ in half-seconds ($\Delta t' = 2\,\Delta\tilde{t}'$, $\Delta t = 2\,\Delta\tilde{t}$) and half-meter sticks ($\Delta x' = 2\,\Delta\tilde{x}'$, $\Delta x = 2\,\Delta\tilde{x}$) and therefore the Lorentz transformation equation would be written as $\Delta\tilde{x} = A\,\Delta\tilde{x}' + B\,\Delta\tilde{t}' + 2\,E\,\Delta\tilde{x}'\,\Delta\tilde{t}'$ which violates the postulate that space is homogeneous. (The astute reader may be asking, "What if we use full seconds and half-meter sticks (i.e. $\Delta t' = \Delta\tilde{t}'$, $\Delta t = \Delta\tilde{t}$, $\Delta x' = 2\,\Delta\tilde{x}'$, $\Delta x = 2\,\Delta\tilde{x}$)?" In such a case, all velocities would also double, so upon substituting the four relations together with $v = 2\,\tilde{v}$ and $c = 2\,\tilde{c}$ we once again regain the same form of the Lorentz transformations.)

The second assumption - that A, B, C, and D are constants - can be justified in various ways. One is that all inertial frames should agree on what “non-accelerating” motion is. That is, if $\Delta x' = u'\,\Delta t'$, then we should also have $\Delta x = u\,\Delta t$, for some constant u. This is true only if the above coefficients are constants. Another justification comes from the second of our two relativity postulates, which says that all points in (empty) space are indistinguishable. With this in mind, let us assume that we have a transformation of the form, say, $\Delta x = A\,\Delta x' + B\,\Delta t' + E\,x'\,\Delta x'$. The x' in the last term implies that the absolute location in space-time (and not just the relative position) is important. Therefore, this last term cannot exist.

Before embarking on actually finding the coefficients in the Lorentz transformation, consider what they would be for the usual Galilean transformation (which are the ones that hold for everyday relative speeds, v). Then we would have $\Delta x = \Delta x' + v\,\Delta t$ and $\Delta t = \Delta t'$ (that is, $A = C = 1$, $B = v$, $D = 0$). We will find, however, under the assumptions of Special Relativity, that this is not the case. The Galilean transformation is not the correct transformation. But we will show below that the correct transformation does indeed reduce to the Galilean transformation in the limit of slow speeds, as it must.

Finding the Lorentz Transformation

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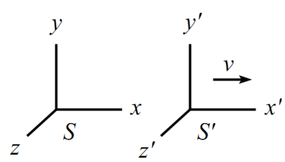

The four constants A, B, C, D are four unknowns, and we can solve for them by using the following four effects:

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Effect	Condition	Result
Time Dilation	$\Delta x'=0$	$\Delta t = \gamma\,\Delta t'$
Length Contraction	$\Delta t'=0$	$\Delta x' = \Delta x/\gamma$
Relative v of Frames	$\Delta x=0$	$\Delta x' = -v\,\Delta t$
Head Start Effect	$\Delta t=0$	$\Delta t' = -v\,\Delta x'/c^2$

Most of the time you will see these equations (and all equations dealing with Lorentz transformations) without the Δ symbols in front. That makes the equations look cleaner, but in such cases it must be understood that x really means Δx and so on. We are always concerned with the *difference* between coordinates of two events in space-time. The actual value of any coordinate is irrelevant, because there is no preferred origin in any frame.

Now would be a great place to pause and make sure that you understand the above four effects and why they lead to their corresponding results. For example the negative sign in the Head Start Effect corresponds to the fact that the rear clock on a train is ahead (i.e. the front clock shows less time). Thus the clock with the higher x' value shows the lower t' value. (In other words, suppose a train is moving to the right with speed v in frame S. Let the first event be "you read the front clock" and the second event be "you read the rear clock." Although in frame S you do these simultaneously ($\Delta t = 0$), in frame S' your actions would be seen as having $\Delta t' < 0$ which, supposing that the clocks are synchronized in frame S', signifies that the person in S first read the front clock and then read the rear clock.)

Substituting in the four conditions and their corresponding results into the Lorentz transformation formula

$$\begin{aligned} \Delta x &= A\,\Delta x' + B\,\Delta t' \\ \Delta t &= C\,\Delta t' + D\,\Delta x' \end{aligned} \tag{5}$$

yields our four constants. Time Dilation yields $C = \gamma$, Length contraction yields $A = \gamma$, Relative v of Frames yields $\frac{B}{A} = v$ and hence $B = \gamma\, v$, and the Head Start Effect yields $\frac{D}{C} = \frac{v}{c^2}$ and hence $D = \frac{\gamma\, v}{c^2}$. Therefore, the Lorentz transformation, in all its glory, is

$$\begin{aligned} \Delta x &= \gamma\,(\Delta x' + v\,\Delta t') \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\,\Delta x'\right) \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \end{aligned} \tag{6}$$

where $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$. We have tacked on the trivial transformations for Δy and Δz, but we won't bother writing these in the future.

We could solve the above equations for the inverse relationships of $\Delta x'$ and $\Delta t'$ in terms of Δx and Δt

$$\begin{aligned} \Delta x' &= \gamma\,(\Delta x - v\,\Delta t) \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\,\Delta x\right) \end{aligned} \tag{7}$$

Of course, these "inverse relationships" simply depend on your point of view. But it's intuitively clear that the only difference between the two sets of equations is the sign of v, because S is simply moving backwards with respect to S'.

The Fundamental Effects

Let us now see how the Lorentz transformations imply the three fundamental effects (namely, the loss of simultaneity, time dilation, and length contraction). Of course, we just used these effects to derive the Lorentz transformation, so we know everything will work out. We'll just be going in circles. But since these fundamental effects are, well, fundamental, let's belabor the point and discuss them one more time, with the starting point being the Lorentz transformations.

- Loss of Simultaneity

Let two events occur simultaneously in frame S'. Then the separation between them, as measured by S', is $(\Delta x', \Delta t') = (\Delta x', 0)$. Using the second of Equation (6), we see that the time between the events, as measured by S, is $\Delta t = \frac{\gamma\, v\, \Delta x'}{c^2}$. This is not equal to zero (unless $x' = 0$). Therefore, the events do not occur simultaneously in frame S.

- Time Dilation

Consider two events that occur in the same place in S'. Then the separation between them is $(\Delta x', \Delta t') = (0, \Delta t')$. Using the second of Equation (6), we see that the time between the events, as measured by S, is

$$\Delta t = \gamma\,\Delta t' \qquad (\text{if } \Delta x' = 0) \tag{8}$$

The factor γ is greater than or equal to 1, so $\Delta t \geq \Delta t'$. The passing of one second on S''s clock takes more than one second on S's clock. S sees S' drinking his coffee very slowly.

The same strategy works if we interchange S and S'. Consider two events that occur in the same place in S. The separation between them is $(\Delta x, \Delta t) = (0, \Delta t)$. Using the second of Equation (7), we see that the time between the events, as measured by S', is

$$\Delta t' = \gamma\, \Delta t \qquad (\text{if } \Delta x = 0) \tag{9}$$

Therefore, $\Delta t' \geq \Delta t$.

Note: If we write down the two above equations by themselves, $\Delta t = \gamma\, \Delta t'$ and $\Delta t' = \gamma\, \Delta t$, they appear to contradict each other. This apparent contradiction arises because we usually omit the conditions they are based on. The former equation is based on the assumption that $\Delta x' = 0$. The latter equation is based on the assumption that $\Delta x = 0$.

- Length Contraction

This proceeds just like the time dilation above, except that now we want to set certain time intervals equal to zero, instead of certain space intervals. We want to do this because to measure a length, we simply measure the distance between two points whose positions are measured simultaneously. That's what a length is.

Consider a stick at rest in S', where it has length l'. We want to find the length l in S. Simultaneous measurements of the coordinates of the ends of the stick in S yield a separation of $(\Delta x, \Delta t) = (\Delta x, 0)$. Using the first of Equation (7), we have

$$\Delta x' = \gamma\, \Delta x \qquad (\text{if } \Delta t = 0) \tag{10}$$

But Δx is by definition the length in S. And $\Delta x'$ is the length in S', because the stick is not moving in S'. Therefore, $l = \frac{l'}{\gamma}$. Since $\gamma \geq 1$, we have $l \leq l'$, so S sees the stick shorter than S' sees it.

Now interchange S and S'. Consider a stick at rest in S, where it has length l. We want to find the length in S'. Measurements of the coordinates of the ends of the stick in S' yield a separation of $(\Delta x', \Delta t') = (\Delta x', 0)$. Using the first of Equation (6), we have

$$\Delta x = \gamma\, \Delta x' \qquad (\text{if } \Delta t' = 0) \tag{11}$$

But $\Delta x'$ is by definition the length in S'. And Δx is the length in S, because the stick is not moving in S. Therefore, $l' = \frac{l}{\gamma}$, so $l' \leq l$.

Note: As with time dilation, if we write down the two above equations by themselves, $l = \frac{l'}{\gamma}$ and $l' = \frac{l}{\gamma}$, they appear to contradict each other. But as before, this apparent contradiction arises from the omission of the conditions they are based on. The former equation is based on the assumptions that $\Delta t = 0$ and that the stick is at rest in S'. The latter equation is based on the assumptions that $\Delta t' = 0$ and that the stick is at rest in S. They have nothing to do with each other.

A Further Note: Here is a wrong way to try and pull out length contraction from the Lorentz transformation. Suppose a stick is at rest in S', so that you measure its length to be $(\Delta x', \Delta t') = (\Delta x', 0)$. What is the length in S? The first of Equation (6) suggests that it is $\Delta x = \gamma\, \Delta x'$, but that is the wrong relation (it predicts that the stick will expand rather than contract). What went wrong? Using the second of Equation (6), we find $\Delta t = \frac{\gamma\, v\, \Delta x'}{c^2}$, which shows that the measurement in frame S was not done at the same time, which is why the correct length of the stick was not measured. This highlights the subtlety required when using the Lorentz transformation - you must ensure that you use the right assumptions for your setup. Rather than using the Lorentz transformation, it is often easier to double check your answer by merely considering Length Contraction, Time Dilation, and the Head Start Effect.

Comments on the Lorentz Transformation

1. The plus or minus sign in the Lorentz transformations corresponds to how the coordinate system on the left-hand side sees the coordinate system on the right-hand side. **If you ever get confused** about how they are related for two frames A and B, simply write down $\Delta x_A = \gamma\,(\Delta x_B \pm v\, \Delta t_B)$ and then imagine sitting in system A and looking at a fixed point in system B. This point satisfies $\Delta x_B = 0$ so that $\Delta x_A = \pm \gamma\, v\, \Delta t_B$. If the point moves to the right (i.e. if Δx_A increases with time) then pick the "+" sign; if it moves to the left, pick the "−" sign.

2. In the limiting case $v \ll c$, the Lorentz transformation equations reduce to

$$\Delta x = \Delta x' + v\,\Delta t + O\left[\tfrac{v}{c}\right]^2 \qquad (12)$$
$$\Delta t = \Delta t' + O\left[\tfrac{v}{c}\right]^2$$

which as discussed above is the Galilean transformation. This must be the case, because we know from everyday experience (where $v \ll c$) that the Galilean transformations work just fine.

3. Often times, you will see the Lorentz transformation written in the more symmetric form

$$\Delta x = \gamma\,(\Delta x' + \beta\,(c\,\Delta t')) \qquad (13)$$
$$c\,\Delta t = \gamma\,((c\,\Delta t') + \beta\,\Delta x')$$

where $\beta = \frac{v}{c}$. The Lorentz transformation can also be written as a symmetric matrix

$$\begin{pmatrix} \Delta x \\ c\,\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \Delta x' \\ c\,\Delta t' \end{pmatrix} \qquad (14)$$

4. One important point is that we must check is that two successive Lorentz transformations (from S_1 to S_2 and then from S_2 to S_3) again yield a Lorentz transformation (from S_1 to S_3). This must be true because we showed that any two frames must be related by the Lorentz transformation equations. If we composed two Lorentz transformations and found that the transformation from S_1 to S_3 was not a Lorentz transformation for some new v, then the whole theory would be inconsistent, and we would have to drop one of our postulates. We can easily prove that the composition of two Lorentz transformations is a two Lorentz transformation using *Mathematica* and the above matrix form.

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ℒ1 = {{1/Sqrt[1 - β1^2], β1/Sqrt[1 - β1^2]}, {β1/Sqrt[1 - β1^2], 1/Sqrt[1 - β1^2]}};
ℒ2 = {{1/Sqrt[1 - β2^2], β2/Sqrt[1 - β2^2]}, {β2/Sqrt[1 - β2^2], 1/Sqrt[1 - β2^2]}};
ℒfin = {{1/Sqrt[1 - β^2], β/Sqrt[1 - β^2]}, {β/Sqrt[1 - β^2], 1/Sqrt[1 - β^2]}};
Solve[ℒfin == Simplify[ℒ1.ℒ2, Assumptions → 0 < β1 < 1&&0 < β2 < 1], β]
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$$\left\{\left\{\beta \to \frac{\beta 1 + \beta 2}{1 + \beta 1\,\beta 2}\right\}\right\}$$

(Recall that $\beta = \frac{v}{c}$.) This shows that after using one Lorentz transformation with velocity v_1 and then another Lorentz transformation with velocity v_2, then the initial and final frames are related by a Lorentz transformation with velocity $v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$. We will derive this result again as a "Velocity Addition Formula" section below.

5. One of the incredible discovery that we can easily check with the Lorentz transformation is that

$$(\Delta x)^2 - c^2 (\Delta t)^2 = (\Delta x')^2 - c^2 (\Delta t')^2 \qquad (15)$$

In other words, subtracting off the square of the time difference (times c) from the square of the distance yields an invariant quantity! As you can imagine, this symmetry is immensely useful in problems.

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Δx = γ (ΔxP + v ΔtP);
Δt = γ (ΔtP + v/c^2 ΔxP);
FullSimplify[-c^2 Δt^2 + Δx^2 /. γ → 1/Sqrt[1 - v^2/c^2]]
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$$-c^2\,\Delta tP^2 + \Delta xP^2$$

Using the Lorentz Transformation

<u>Example</u>
A train with proper length L moves with speed $\frac{5c}{13}$ with respect to the ground. A ball is thrown from the back of the train to the front. The speed of the ball with respect to the train is $\frac{c}{3}$. As viewed by someone on the ground, how much time does the ball spend in the air, and how far does it travel?

<u>Solution</u>
The γ factor associated with the train's speed $\frac{5c}{13}$ is $\gamma = \frac{13}{12}$. The two events we are concerned with are "ball leaving back of train" and "ball arriving at front of train." The space and time separation between these events is easy to calculate in the train's frame, where $\Delta x_T = L$ and $\Delta t_T = \frac{L}{c/3} = \frac{3L}{c}$. The Lorentz transformation giving the coordinates on the ground is

$$\begin{aligned} \Delta x_G &= \gamma\,(\Delta x_T + v\,\Delta t_T) \\ \Delta t_G &= \gamma\left(\Delta t_T + \frac{v\,\Delta x_T}{c^2}\right) \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} \Delta x_G &= \frac{13}{12}\left(L + \frac{5c}{13}\,\frac{3L}{c}\right) = \frac{7L}{3} \\ \Delta t_G &= \frac{13}{12}\left(\frac{3L}{c} + \frac{5c}{13}\,\frac{L}{c^2}\right) = \frac{11L}{3c} \end{aligned} \tag{17}$$

Of course, we could have gotten this answer using our tactics from the previous lectures; the Lorentz transformation contains no new information. For example, we could compute that the time interval seen on the ground would be the time dilation of the time it takes on the train - taking into account the Head Start effect between the two clocks, $\Delta t_G = \gamma\left(\Delta t_T + \frac{L v}{c^2}\right) = \frac{13}{12}\left(\frac{3L}{c} + \frac{L}{c^2}\,\frac{5c}{13}\right) = \frac{11L}{3c}$. After that, the distance traveled by the ball as seen by the ground would equal the length of the contracted train ($\frac{L}{\gamma}$) plus the extra distance traveled by the train while the ball is in the air ($v\,\Delta t_G$) to yield the total length $\Delta x_G = \frac{L}{\gamma} + v\,\Delta t_G = \frac{12L}{13} + \frac{5c}{13}\,\frac{11L}{3c} = \frac{7L}{3}$. There are other ways to compute these values, but all of them are much more painful then solving the problem in the train frame and then plugging and chugging into the Lorentz transformation equations. (In most problems, there will be one frame where the values are extremely easy to compute, so this strategy works brilliantly!) □

Velocity Addition Formula

Complementary Section: Longitudinal Velocity Addition

Consider the following setup. An object moves at speed v_1 with respect to frame S'. And frame S' moves at speed v_2 with respect to frame S, in the same direction as the motion of the object. What is the speed, u, of the object with respect to frame S?

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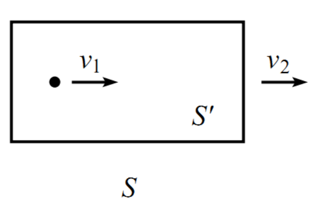

The Lorentz transformation may be used to easily answer this question. The relative speed of the frames is v_2.

Consider two events along the object's path (for example, say it makes some beeps). We are given that $\frac{\Delta x'}{\Delta t'} = v_1$. Our goal is to find $u \equiv \frac{\Delta x}{\Delta t}$. The Lorentz transformation from S' to S is

$$\Delta x = \gamma_2(\Delta x' + v_2\, \Delta t') \tag{18}$$
$$\Delta t = \gamma_2\left(\Delta t' + \frac{v_2\, \Delta x'}{c^2}\right) \tag{19}$$

where $\gamma_2 = \frac{1}{\left(1-\frac{v_2^2}{c^2}\right)^{1/2}}$. Therefore,

$$\begin{aligned} u &\equiv \frac{\Delta x}{\Delta t} \\ &= \frac{\frac{\Delta x'}{\Delta t'}+v_2}{1+\frac{v_2}{c^2}\frac{\Delta x'}{\Delta t'}} \\ &= \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} \end{aligned} \tag{20}$$

This is the *velocity-addition formula* (for adding velocities in the same direction). Let’s look at some of its properties.

- It is symmetric with respect to v_1 and v_2, as it should be, because we could switch the roles of the object and frame S.
- For $v_1\, v_2 \ll c^2$, it reduces to $u \approx v_1 + v_2$, which we know holds perfectly well for everyday speeds.
- If $v_1 = c$ or $v_2 = c$, then $u = c$, as should be the case, because anything that moves with speed c in one frame moves with speed c in another.
- The maximum (or minimum) of u in the region $-c \leq v_1,\, v_2 \leq c$ equals c (or $-c$), which can be seen by noting that the partial derivatives $\frac{\partial u}{\partial v_1} = \frac{1-\left(v_2^2/c^2\right)}{\left(1+\left(v_1\, v_2/c^2\right)\right)^2}$ and $\frac{\partial u}{\partial v_2} = \frac{1-\left(v_1^2/c^2\right)}{\left(1+\left(v_1\, v_2/c^2\right)\right)^2}$ are never zero in the interior of the region.

If you take any two velocities that are less than c, and add them according to the above equation, then you will obtain a velocity that is again less than c. This shows that no matter how much you keep accelerating an object (that is, no matter how many times you give the object a speed v_1 with respect to the frame moving at speed v_2 that it was just in), you can’t bring the speed up to the speed of light. We’ll give another argument for this result when we discuss energy.

Let’s consider when the velocity-addition formula is needed. Consider the following two scenarios.

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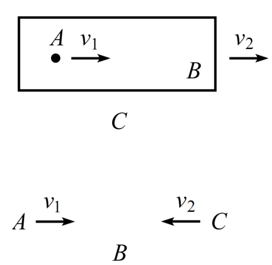

The velocity-addition formula is required in both the upper and lower scenario when our goal is to find the speed of A with respect to C. More generally, the velocity-addition formula applies when we ask, "If A moves at v_1 with respect to B, and B moves at v_2 with respect to C (which means, of course, that C moves at speed v_2 with respect to B), then how fast does A move with respect to C?"

The velocity-addition formula *does not* apply if we ask the more mundane question, "What is the relative speed of A and C, as viewed by B?" The answer to this is simply $v_1 + v_2$ in the bottom scenario, but for the top scenario we would have to transform into B's frame, where C would have velocity v_2 pointing to the left, but to determine $A's$ velocity which would require using the velocity-addition formula.

Complementary Section: The Sign of the Velocity Addition Formula

Equal Speeds

Example

A and B travel at $\frac{4c}{5}$ and $\frac{3c}{5}$, respectively. How fast should C travel between them, so that she sees A and B approaching her at the same speed? What is this speed?

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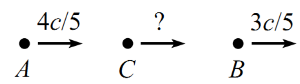

Solution

Method 1: Suppose C is traveling with v, then the A's velocity in C's frame would equal $\frac{\frac{4}{5}c-v}{1-\frac{\frac{4}{5}cv}{c^2}}$. (In other words, this would be what the velocity of A would look like for an observer on a train moving to the right at velocity v.) We have defined positive velocity to the right, so that we are relativistically boosting the velocities to the left by $-v$ (as a check, note that C's final velocity in this boosted frame will be 0).

Similarly, the velocity of B in C's frame point would be $\frac{\frac{3}{5}c-v}{1-\frac{\frac{3}{5}cv}{c^2}}$ pointing to the right, or $\frac{v-\frac{3}{5}c}{1-\frac{\frac{3}{5}cv}{c^2}}$ pointing to the *left*.

Because we want for A and B to both approach C with the same velocity, we set

$$\frac{\frac{4}{5}c-v}{1-\frac{\frac{4}{5}cv}{c^2}} = \frac{v-\frac{3}{5}c}{1-\frac{\frac{3}{5}cv}{c^2}} \tag{21}$$

which we solve using *Mathematica*

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Solve[(4/5 c - v)/(1 - (4/5 c v)/c^2) == -((3/5 c - v)/(1 - (3/5 c v)/c^2)), v]
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{{v -> 5 c/7}, {v -> 7 c/5}}
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We find the two solutions $v = \frac{5}{7}c$, $\frac{7}{5}c$; since $v = \frac{7}{5}c$ implies traveling faster than the speed of light, the only physical solution is $v = \frac{5}{7}c$. The velocities of A or B in C's frame are found by substituting $v = \frac{5}{7}c$ above, yielding $\frac{\frac{4}{5}c-\frac{5}{7}c}{1-\frac{\left(\frac{4}{5}c\right)\left(\frac{5}{7}c\right)}{c^2}} = \frac{c}{5}$ so that A and B travel at speed $\frac{c}{5}$ towards C.

Method 2: Using $v_A = \frac{4}{5}c$ and $v_B = \frac{3}{5}c$, the relative speed of A in B's frame equals $\frac{\frac{4}{5}c-\frac{3}{5}c}{1-\frac{\left(\frac{4}{5}c\right)\left(\frac{3}{5}c\right)}{c^2}} = \frac{5}{13}c$. From C's point of view (where A and B are approaching each other at speed $\tilde{v}$), this $\frac{5}{13}c$ is the result of relativistically adding $\tilde{v}$ with another $\tilde{v}$, which implies $\frac{5}{13}c = \frac{2\tilde{v}}{1+\frac{\tilde{v}^2}{c^2}}$. Solving this yields $\tilde{v} = \frac{1}{5}c$, $5c$. Only the former is physical, so that A and B approach each other in C's frame at speed $\frac{c}{5}$. We could then use the equation $\frac{\frac{4}{5}c-v}{1-\frac{\frac{4}{5}cv}{c^2}} = \frac{c}{5}$ to solve for C's value $v = \frac{5}{7}c$, as found above. □

Extra Problem: Fizeau Experiment

Example

The second postulate of relativity says that the speed of light in vacuum is always c (in an inertial frame). However, the speed of light in a medium (such as water) is given by $\frac{c}{n}$, where n is the *index of refraction* of the medium. For water, n is about $\frac{4}{3}$.

Imagine aiming a beam of light rightward into a pipe of water moving rightward with speed v. Naively, the speed of the light with respect to the ground should be $\frac{c}{n} + v$. Find the correct speed using the velocity addition formula. Then, in the case where $v \ll c$ (which is certainly a valid approximation in the case of moving water), show that to leading order in v, the speed takes the form of $\frac{c}{n} + A\,v$. What is the value of A?

Solution

Since the light moves at speed $\frac{c}{n}$ with respect to the water, rand the water moves at speed v with respect to the ground, the velocity addition formula gives the speed of the light with respect to the ground as

$$V = \frac{\frac{c}{n}+v}{1+\frac{\left(\frac{c}{n}\right)(v)}{c^2}} = \frac{\frac{c}{n}+v}{1+\frac{v}{n\,c}} \tag{22}$$

Taking the first order Taylor approximation in $v \ll c$, we obtain the expression

$$V \approx \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) v \tag{23}$$

so that $A = 1 - \frac{1}{n^2}$.

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Series[(c/n + v)/(1 + v/(n c)), {v, 0, 1}]
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$$\frac{c}{n} + \left(1 - \frac{1}{n^2}\right) v + O[v]^2$$

We can check this result in a few special cases. If $n = 1$, which means that we have a vacuum instead of water, we obtain a speed of $V = c$. This is correct because we know that light always moves with speed c in vacuum. If n is very large (implying a very dense medium), we obtain a speed of $V \approx \frac{c}{n} + v$. This is correct because it is the naive addition of the speed,s which we know works perfectly fine when both speeds are much less than c.

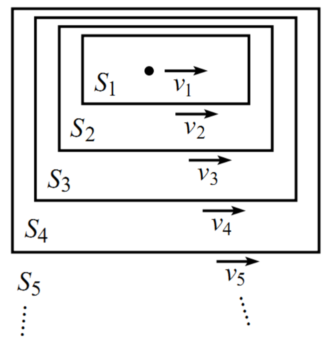
In 1851, well before Einstein's velocity addition formulas was known, Fizeau performed an experiment to measure the speed (with respect to the ground) of light in moving water. His setup involved an interferometer similar to the one Michelson and Morley used in their experiment. He obtained a result consistent with the approximate formula Equation (23), so he conjectured that the formula held (exactly) in general. Many people then made unsuccessful attempts (involving frame dragging of the "ether," for example (this was before it was known that there is no ether)) to explain why the parameter A took on the value $1 - \frac{1}{n^2}$ instead of the naive value of 1. In retrospect, of course, failure was the likely result of their (commendable) efforts to generate an exact theory from an approximate result. It was more than half a century until Einstein produced the theory of special relativity in 1905, from which the correct explanation of A's value followed via the velocity addition formula (along with the approximation we made in Equation (23)). Conversely, the result of Fizeau's experiment was highly influential in Einstein's formulation of special relativity. □

Many Velocity Additions

Example

An object moves at speed $\beta_1 = \frac{v_1}{c}$ with respect to S_1, which moves at speed β_2 with respect to S_2, which moves at speed β_3 with respect to S_3, and so on, until finally S_{N-1} moves at speed β_{N-1} with respect to S_N.

Out[ ]=



Show that the speed, $\beta_{(N)}$, of the object with respect to S_N can be written as

$$\beta_{(N)} = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-} \tag{24}$$

where

$$P_N^+ = \Pi_{j=1}^N (1 + \beta_j) \tag{25}$$

$$P_N^- = \Pi_{j=1}^N (1 - \beta_j) \tag{26}$$

Solution

We proceed by induction. When $N = 1$, $\beta_{(N)} = \beta_1$ as expected. Now assume that the formula holds for N, and let us show that it holds for $N+1$ as well. Using the velocity addition formula, the velocity in the $N+1$ frame equals

$$\begin{aligned}\frac{\beta_{N+1} + \beta_{(N)}}{1 + \beta_{N+1}\,\beta_{(N)}} &= \frac{\beta_{N+1} + \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}}{1 + \beta_{N+1}\,\frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}} \\ &= \frac{\beta_{N+1}(P_N^+ + P_N^-) + (P_N^+ - P_N^-)}{(P_N^+ + P_N^-) + \beta_{N+1}(P_N^+ - P_N^-)} \\ &= \frac{P_N^+(1+\beta_{N+1}) - P_N^-(1-\beta_{N+1})}{P_N^+(1+\beta_{N+1}) + P_N^-(1-\beta_{N+1})} \\ &= \frac{P_{N+1}^+ - P_{N+1}^-}{P_{N+1}^+ + P_{N+1}^-} \\ &= \beta_{(N+1)}\end{aligned} \tag{27}$$

as desired. This proves that the result holds for all N.

Notice that $\beta_{(N)}$ is symmetric in the β_j's, as we expect from the symmetry of the velocity addition formula (in other words, it doesn't matter the order in which the β_j are applied). Additionally, if at least one of the β_j equal 1, then $P_N^- = 0$ and $\beta_{(N)} = 1$, as it should. Similarly, if at least one of the β_j equal -1, then $P_N^+ = 0$ so that $\beta_{(N)} = -1$, as it should. (If one of the β_j equals 1 while another equals -1, then we get the indeterminate $\frac{0}{0}$. This is caused by the fact that depending on how you approach this limit, you could get different answers; this same indeterminate answer is seen in the normal velocity addition formula when $v_1 = c$ and $v_2 = -c$.)

To get a sense for this effect, suppose that all of the $\beta_j = \frac{1}{2}$ for all j, then the first few resulting β's would be

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Table[(Product[1 + β, {nn}] - Product[1 - β, {nn}]) / (Product[1 + β, {nn}] + Product[1 - β, {nn}]), {nn, 1, 6}] /. β → 0.5
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{0.5, 0.8, 0.928571, 0.97561, 0.991803, 0.99726}
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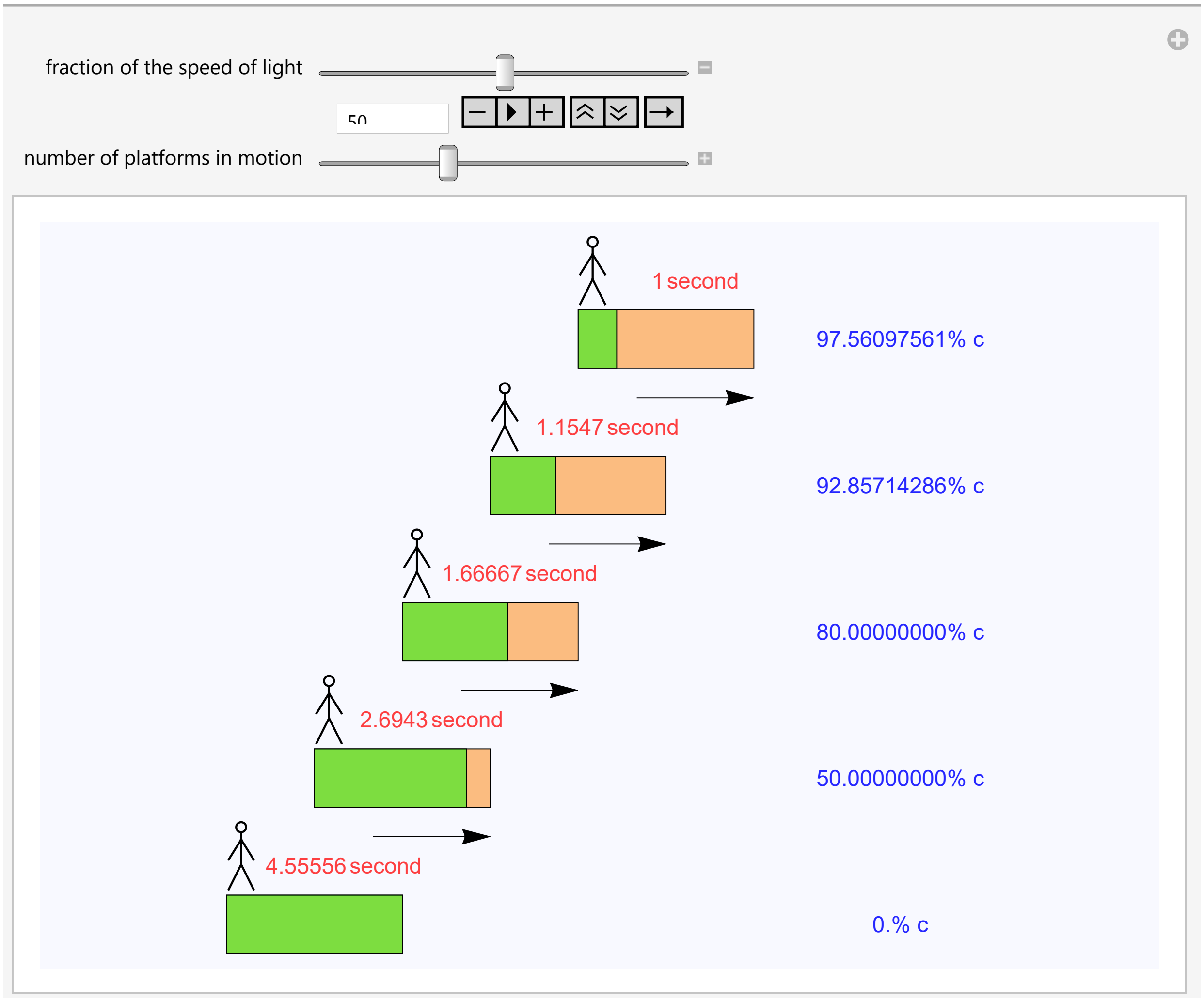
The following diagram lets you visualize this effect for the case where all β_j's have the same value, but you can manipulate what that value is (using the top slide in *Mathematica*). You can also choose vary the number of platforms using the bottom slides. Green shows the length contraction of each platform (in the bottom platform's frame), red shows the time dilation effect (in the bottom platform's frame when the top platform measures 1 second), and blue indicates the percent of the speed of light at which each person is traveling.

Out[ ]=



There are several interesting points to notice. First, no matter how fast the $0 \leq \beta < 1$ value is, none of the platforms can ever reach the speed of light. Second, when $\beta \approx 0$ we see the standard addition of velocities, with the velocities of the platforms from the bottom up taking the values 0, β, 2β, 3β ...; as β is increased, this linear trend starts to break down as subsequently adding another β results in a change that is smaller than β. As your speed approaches the speed of light, adding β leads to a negligible change. □

Visualizing Special Relativity

Find special relativity confusing? Want to really "see" the effects with your eyes? Velocity Raptor has got your back (consider it as *optional research*). In this game, you must use the powers given to you by Special Relativity to fight the forces of evil.

While I encourage you to play this game in its entirety (it takes about 30 minutes), I urge you to at least try to achieve level 26. For in playing level 25 and level 26, you will how you would *physically observe with your eyes* the world at velocities near the speed of light. And once you see this, never use that reasoning in this class. Because we don’t care about when photons actually hit your eyes; we describe "seeing" in Special Relativity as describing what actually happens in a frame. (In other words, we use "seeing" to describe a room full of synchronized clocks in level 18, whereas what you would *physically observe with your eyes* (which we don’t care about!) is shown in level 26).